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Non-Abelian charge quantisation and the Bohr–Wilson–Sommerfeld condition

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Abstract. The quantisation of the non-Abelian charge of a point-like particle is shown to arise from the BWS condition in the pre-quantisation scheme of Kostant and Souriau.

1. Introduction

In generalised Kaluza-Klein theories one assumes a spontaneous fibration of a higher-dimensional space B in a local product of ordinary spacetime M with a compact 'internal' N-dimensional fibre F. Usually F is taken to be a Lie group G or a right coset space H\G. The fibration is spontaneous in the sense that the field equations for the metric on B have a 'ground-state' solution given by the direct sum of a metric on M and a G-invariant metric on F. Thus, the generators of G define Killing vectors on B and a right action of G. Such metrics incorporate Yang-Mills potentials of a subgroup of G and generalise the five-dimensional Kaluza metric to higher dimensions. From the Einstein equations on B, one obtains, for the generalised Kaluza-Klein metric, field equations which lead to a theory of coupled Einstein, Yang-Mills and scalar fields in four dimensions; this 'dimensional reduction' can be obtained either by working in a frame where the Kaluza-Klein metric becomes independent of the internal coordinates, or by performing some averaging procedure over the fibres†.

If matter fields or wavefunctions on B are also considered and subjected to invariant wave equations, following Klein's proposal, one finds that such fields describe in four dimensions particles with quantised charges and masses. In particular, Klein's method can be applied to a wavefunction Ψ in $4+N$ dimensions, subjected to the invariant equation

$$[\Delta+(mc/h)^2]\Psi(X)=0, \tag{1.1}$$

where Δ is the Laplace operator on B with the Kaluza-Klein metric. In Klein's paper, one had $N=1$ and $m=0$, leading to a U(1) bundle for B and to the quantisation rules $q_n=n(l/a)\varepsilon$, $m_n=|n|\hbar/ac$, where $n=0, \pm1, \pm2, \ldots$, l is the Planck length, a is the length of the internal fibre and ε is the electron charge. In such a framework Weinberg (1984) has given a prescription for calculating the gauge coupling constants in the general case.

† For all additional details and references to the original literature see Cho (1975), Cho and Freund (1975), Orzalesi (1981), Witten (1981) and Salam and Strathdee (1982). For more mathematically oriented remarks see Coqueraux and Jadczyk (1983) and Coqueraux (1983).

In this paper, it is shown that a quantisation of charge also emerges within a different framework, which is pre-quantum mechanical: namely we show that the prequantisation scheme of Kostant and Souriau implies the quantisation of the charges. Closely related to our work is that of Duval (1981) and Duval and Horvathy (1982)†, the main difference being that we emphasise the realisation of the Poisson algebra of the observables in the Hilbert space of square integrable functions on phase space (van Hove 1951).

In this framework, matter is described by (classical) point-like particles which move along time-like geodesics on B and, in order to interpret the non-Abelian charges as components of the momentum in $4+N$ dimensions, it is assumed that the fibre F is the gauge group G itself, so that B is a principal fibre bundle over spacetime M with structure group G. The gauge group G is taken semi-simple and compact.

In § 2 the well known equations of motion for such dynamical systems are given (Cho 1975, Orzalesi and Pauri 1982, Vanhecke 1982) and in § 3 the Hamiltonian formalism and the reduction of phase space due to the presence of symmetry are discussed. In the last section it is shown how the Kostant–Souriau pre-quantisation scheme (Woodhouse 1980) implies the quantisation of the charges. Some formulae on the non-holonomic formulation of the Lagrangian and Hamiltonian formalism are given in appendix 1; the mentioned pre-quantisation scheme is briefly outlined in appendix 2 and in appendix 3 some comments are made on the ansatz for the $4+N$ dimensional metric and on the 'spontaneous fibration' aspect of the theory.

2. The equations of motion

Let B be a principal fibre bundle over spacetime M with projection $\pi: B \to M$ and structure group G. In a local gauge ϕ the points $X, Y, \ldots$ of B are given by $X = \phi(x, \xi)$, where $x = \pi(X)$ is a point of M with coordinates x^i ($i = 0, 1, 2, 3$) and where ξ is a group element with dimensionless canonical coordinates ξ^α ($\alpha = 1, 2, \ldots, N$) defined by $\xi = \exp(\xi^\alpha T_\alpha)$. The generators T_α of the Lie algebra $\mathscr{L}(\mathrm{G})$ obey

$$[T_\alpha, T_\beta] = f^\gamma_{\alpha\beta} T_\gamma \tag{2.1}$$

and, since G is assumed to be semi-simple and compact, the Killing form

$$\eta_{\alpha\beta} = f^\nu_{\alpha\mu} f^\mu_{\beta\nu} \tag{2.2}$$

is non-degenerate and negative definite. The group G acts globally on the right on B: $(\gamma \in \mathrm{G})$

$$R_\gamma : B \to B : X = \phi(x, \xi) \to R_\gamma(X) = \phi(x, \xi \cdot \gamma) = X \cdot \gamma. \tag{2.3}$$

This action is generated by the fundamental vector fields

$$e_\alpha(X) = \frac{\mathrm{d}}{\mathrm{d}t}(X \cdot \gamma_\alpha(t))\bigg|_{t=0} \tag{2.4}$$

where

$$\gamma_\alpha(t) = \exp(tT_\alpha). \tag{2.5}$$

† I thank the referee for drawing my attention to this work.

The bundle is endowed with a connection defined by the 1-forms $\boldsymbol{\omega}^\alpha(X)$, given in a local gauge by

$$\boldsymbol{\omega}^\alpha(x,\xi)=\mathrm{Ad}^\alpha{}_\beta(\xi^{-1})A_i^\beta(x)\,\mathrm{d}\boldsymbol{x}^i+L^\alpha{}_\beta(\xi^{-1},\xi)\,\mathrm{d}\boldsymbol{\xi}^\beta, \tag{2.6a}$$

where $A_i^\beta(x)$ are the gauge potentials and where

$$L^\alpha{}_\beta(\xi,\zeta)=\partial(\xi\cdot\zeta)^\alpha/\partial\zeta^\beta \qquad R^\alpha{}_\beta(\xi,\zeta)=\partial(\zeta\cdot\xi)^\alpha/\partial\zeta^\beta$$

define the left and right auxiliary matrices such that

$$\mathrm{Ad}(\xi)=L(\xi,\xi^{-1})R(\xi^{-1},1)=R(\xi^{-1},\xi)L(\xi,1)$$

is the $N\times N$ adjoint representation of G in $\mathscr{L}(\mathrm{G})$. These 1-forms together with

$$\boldsymbol{\omega}^i(x,\xi)=\mathrm{d}\boldsymbol{x}^i \tag{2.7}$$

form a basis of cotangent space T_X^*B with dual

$$\boldsymbol{e}_\alpha(x,\xi)=(\boldsymbol{\partial}/\partial\xi^\beta)L^\beta{}_\alpha(\xi,1) \tag{2.8a}$$

$$\boldsymbol{e}_i(x,\xi)=\boldsymbol{\partial}/\partial x^i-(\boldsymbol{\partial}/\partial\xi^\beta)R^\beta{}_\alpha(\xi,1)A_i^\alpha(x). \tag{2.8b}$$

The structure functions of the anholonomic basis are given by

$$\mathrm{d}\boldsymbol{\omega}^i=0 \tag{2.9a}$$

$$\mathrm{d}\boldsymbol{\omega}^\alpha=-\tfrac{1}{2}f^\alpha_{\beta\gamma}\boldsymbol{\omega}^\beta\wedge\boldsymbol{\omega}^\gamma+\tfrac{1}{2}\Omega_{ij}{}^\alpha(X)\boldsymbol{\omega}^i\wedge\boldsymbol{\omega}^j \tag{2.9b}$$

where the curvature is given in a local gauge by

$$\Omega_{ij}{}^\alpha(x,\xi)=\mathrm{Ad}^\alpha{}_\beta(\xi^{-1})F_{ij}{}^\beta(x) \tag{2.10}$$

with field strengths

$$F_{ij}{}^\alpha=\partial_iA_j^\alpha-\partial_jA_i^\alpha+f^\alpha_{\beta\gamma}A_i^\beta A_j^\gamma. \tag{2.11}$$

The non-Abelian Kaluza–Klein metric in bundle space is given by (Cho 1975)†

$$\mathscr{g}(X)=g_{ij}(\pi(X))\boldsymbol{\omega}^i\otimes\boldsymbol{\omega}^j+a^2\eta_{\alpha\beta}\boldsymbol{\omega}^\alpha\otimes\boldsymbol{\omega}^\beta \tag{2.12}$$

where a is a constant with the dimensions of length. Its Riemannian curvature scalar, constant on each fibre, reads

$$\mathscr{R}(X)=R(x)+N/4a^2+\tfrac{1}{4}a^2\eta_{\alpha\beta}F_{ij}{}^\alpha(x)F^{ij\beta}(x) \tag{2.13}$$

where $R(x)$ is the Riemannian curvature scalar of the metric $g_{ij}(x)$ in spacetime M.

The field action is taken as

$$S_{\text{field}}=\frac{c^3}{16\pi\kappa}\int\mathrm{d}^4x|\det g(x)|^{1/2}(\mathscr{R}(x)-2\lambda) \tag{2.14}$$

where κ is the gravitational coupling constant, c is the speed of light and λ is a cosmological constant introduced to eventually cancel with the group curvature $N/4a^2$ appearing in (1.13).

The matter is described by point-like particles with a typical trajectory $Z(u)$—locally $z^i(u)$, $\zeta^\alpha(u)$—where u is an evolution parameter. The action is proportional to the

† The implementability of the spontaneous fibration and this ansatz for the metric are discussed in appendix 3.

$(4+N)$-dimensional arc length:

$$S_{\text{matter}} = -mc \int \mathrm{d}\sigma \tag{2.15}$$

with

$$\mathrm{d}\sigma/\mathrm{d}u = \dot{\sigma} = |g_Z[\dot{\boldsymbol{Z}}, \dot{\boldsymbol{Z}}]|^{1/2} \tag{2.16}$$

where $\dot{\boldsymbol{Z}} = \mathrm{d}Z/\mathrm{d}u$ is the velocity. The trajectories of the particles are the geodesics in bundle space (Orzalesi and Pauri 1982, Vanhecke 1982).

The explicit form of the equations of motion is most easily obtained from the formulae of appendix 1 using the non-holonomic components of the velocities:

$$v^a = (\boldsymbol{\omega}^a, \dot{\boldsymbol{Z}}) \qquad a = i, \alpha \tag{2.17}$$

and the Lagrangian

$$L = -mc(g_{ij}(z)v^iv^j + a^2\eta_{\alpha\beta}v^\alpha v^\beta)^{1/2}. \tag{2.18}$$

The momentum $\boldsymbol{p}$ is defined by:

$$(\boldsymbol{p}, \boldsymbol{w}) = \frac{\mathrm{d}}{\mathrm{d}t} L(Z, \boldsymbol{v} + t\boldsymbol{w})|_{t=0} \tag{2.19}$$

and its holonomic $p_\alpha = (\boldsymbol{p}, \boldsymbol{\partial}/\partial Z^\alpha)$ and non-holonomic components $r_a = (\boldsymbol{p}, \boldsymbol{e}_a) = \partial L/\partial v^a$ are related by

$$r_i = p_i - p_\alpha R^\alpha{}_\beta(\zeta, 1)A_i^\beta(z) \tag{2.20a}$$

$$r_\alpha = p_\beta L^\beta{}_\alpha(\zeta, 1). \tag{2.20b}$$

In terms of the velocities, the non-holonomic components read:

$$r_i = -(mc/\dot{\sigma})g_{ij}(z)v^j \tag{2.21a}$$

$$r_\alpha = -(mc/\dot{\sigma})a^2\eta_{\alpha\beta}v^\beta. \tag{2.21b}$$

The geodesic equations are

$$mc\left[\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{v^i}{\dot{\sigma}}\right) + \begin{Bmatrix} i \\ pq \end{Bmatrix}\frac{v^pv^q}{\dot{\sigma}}\right] = -r_\alpha\ \mathrm{Ad}^\alpha{}_\beta(\zeta^{-1})F^i{}_j{}^\beta(z)v^j \tag{2.22a}$$

$$\mathrm{d}r_\alpha/\mathrm{d}u = 0. \tag{2.22b}$$

In order to relate these equations in an unambiguous way to the non-Abelian Lorentz equation and charge conservation it is necessary to consider the field equations

$$R_{ij} - \tfrac{1}{2}Rg_{ij} + (\lambda - N/8a^2)g_{ij} + (8\pi\kappa/c^3)T_{ij} = 0 \tag{2.23a}$$

$$\nabla_i\mathcal{F}^{ij\alpha} = (4\pi/c)J^{j\alpha}, \tag{2.23b}$$

where we have introduced the field strengths

$$\mathcal{F}_{ij}{}^\alpha = eF_{ij}{}^\alpha \tag{2.24}$$

with the universal† unit of charge

$$e = ac^2/2\sqrt{\kappa}. \tag{2.25}$$

† The charge e is 'universal' in the sense that it does not depend on the properties of a specific particle, but only on the universal constants c and κ and on the size a of the fibre.

This unit of charge is introduced so that the Yang–Mills part of the energy-momentum tensor in (2.23a) reads

$$T_{ij}^{\mathrm{YM}} = -(1/4\pi c)(\tfrac{1}{4}g_{ij}\mathcal{F}_{pq}{}^{\alpha}\mathcal{F}^{pq\beta} - \mathcal{F}_{ip}{}^{\alpha}\mathcal{F}_{j}{}^{p\beta})\eta_{\alpha\beta}. \tag{2.26}$$

The matter contribution to the energy-momentum tensor is as usual:

$$T_{ij}^{\mathrm{matter}}(x) = |\det g(x)|^{-1/2} \int \mathrm{d}u\, mc \frac{v_i v_j}{\dot{\sigma}} \delta^4(x - z(u)). \tag{2.27}$$

The derivative in (2.23b) is a doubly covariant derivative and the current is expressed as:

$$J^{j\alpha}(x) = |\det g(x)|^{-1/2} \int \mathrm{d}u\, cI^{\alpha}(u) v^j \delta^4(x - z(u)) \tag{2.28}$$

with

$$I^{\alpha}(u) = \mathrm{Ad}^{\alpha}{}_{\beta}(\zeta(u)) Q^{\beta}(u) \tag{2.29a}$$

$$Q^{\alpha}(u) = e\frac{mc^2}{e^2/a} k^{\alpha}(u) \tag{2.29b}$$

$$k^{\alpha}(u) = -av^{\alpha}/\dot{\sigma}. \tag{2.29c}$$

The 'intrinsic' or 'body-fixed' charge Q^{α} is related to the momentum by

$$r_{\alpha} = \eta_{\alpha\beta}(e/c)Q^{\beta}. \tag{2.30}$$

From the equations of motion it follows that the dimensionless k^{α} are constant so that along the trajectory

$$\mathrm{d}\sigma^2 = (1 - \eta_{\alpha\beta}k^{\alpha}k^{\beta})^{-1}\,\mathrm{d}s^2 \tag{2.31}$$

and since $\eta_{\alpha\beta}$ is negative definite $\mathrm{d}\sigma^2$ and $\mathrm{d}s^2$ have the same sign. We may thus choose the four-dimensional arc length as evolution parameter with

$$\dot{\sigma} = (1 - \eta_{\alpha\beta}k^{\alpha}k^{\beta})^{-1/2}. \tag{2.32}$$

The equations of motion are finally written as

$$m'c^2\left(\frac{\mathrm{d}}{\mathrm{d}s}v^i + \left\{ {i \atop pq} \right\} v^p v^q\right) = -\eta_{\alpha\beta} I^{\alpha}\mathcal{F}^{i}{}_{j}{}^{\beta}(z)v^j \tag{2.33a}$$

and

$$\mathrm{d}Q^{\alpha}/\mathrm{d}s = 0 \tag{2.33b}$$

or equivalently

$$\mathrm{d}I^{\alpha}/\mathrm{d}s + f^{\alpha}_{\beta\gamma}A_i^{\beta}(z)v^i I^{\gamma} = 0. \tag{2.33c}$$

It should be noticed that the 'dressed' mass m',

$$m' = m(1 - \eta_{\alpha\beta}k^{\alpha}k^{\beta})^{1/2} \tag{2.34}$$

also appears in the expression of the energy-momentum tensor

$$T_{ij}^{\mathrm{matter}}(x) = |\det g(x)|^{-1/2} \int \mathrm{d}s\, m'cv_i v_j \delta^4(x - z(s)).$$

Formally I^α can be obtained as an ordered integral along the spacetime trajectory of the 1-form $T_\alpha A_i^{\ \alpha}(x)\,\mathrm{d}\boldsymbol{x}^i$. Substitution of the obtained result in (2.33a) leads to a formal integro-differential equation (Vanhecke 1982).

3. Hamiltonian formalism—reduction of phase space

We limit ourselves to study the motion of a test particle in a fixed field configuration so that the configuration space of the system is the bundle space B itself. While the Lagrangian formalism is built upon the tangent bundle TB, with typical points $(Z, \boldsymbol{v})$, the Hamiltonian approach is defined in phase space which is the cotangent bundle T^*B with points $P=(Z, \boldsymbol{p})$†. Canonical and non-canonical coordinates of points in phase space are given by $(z^i, \zeta^\alpha, p_i, p_\alpha)$ and $(z^i, \zeta^\alpha, r_i, r_\alpha)$. The holonomic basis of $T_P(T^*B)$ associated with the non-canonical coordinates is formed by the vector fields‡

$$|\partial/\partial z^i\rangle \qquad |\partial/\partial \zeta^\alpha\rangle \qquad |\partial/\partial r_i\rangle \qquad |\partial/\partial r_\alpha\rangle. \tag{3.1}$$

It will be more convenient to use the anholonomic basis

$$|e_i\rangle = |\partial/\partial z^i\rangle - |\partial/\partial \zeta^\alpha\rangle R^\alpha{}_\beta(\zeta, 1)A_i^{\ \beta}(z) \tag{3.2a}$$

$$|e_\alpha\rangle = |\partial/\partial \zeta^\beta\rangle L^\beta{}_\alpha(\zeta, 1) \tag{3.2b}$$

$$|\partial/\partial r_i\rangle \qquad |\partial/\partial r_\alpha\rangle. \tag{3.2c}$$

The corresponding dual bases of $T_P^*(T^*B)$ are

$$\langle \mathrm{d}z^i| \qquad \langle \mathrm{d}\zeta^\alpha| \qquad \langle \mathrm{d}r_i| \qquad \langle \mathrm{d}r_\alpha| \tag{3.3}$$

and

$$\langle \omega^i| = \langle \mathrm{d}z^i| \tag{3.4a}$$

$$\langle \omega^\alpha| = \mathrm{Ad}^\alpha{}_\beta(\zeta^{-1})A_i^{\ \beta}(z)\langle \mathrm{d}z^i| + L^\alpha{}_\beta(\zeta^{-1}, \zeta)\langle \mathrm{d}\zeta^\beta| \tag{3.4b}$$

$$\langle \mathrm{d}r_i| \qquad \langle \mathrm{d}r_\alpha|. \tag{3.4c}$$

The canonical symplectic potential on phase space reads

$$\langle\theta| = r_i\langle\omega^i| + r_\alpha\langle\omega^\alpha| \tag{3.5}$$

and the symplectic 2-form is given by

$$\begin{aligned}\Sigma = -\mathrm{d}\langle\theta| &= \langle\omega^i|\wedge\langle\mathrm{d}r_i| + \langle\omega^\alpha|\wedge\langle\mathrm{d}r_\alpha| \\ &\quad -\tfrac{1}{2}r_\alpha\Omega_{ij}^{\ \ \alpha}(Z)\langle\omega^i|\wedge\langle\omega^j| + \tfrac{1}{2}r_\alpha f^\alpha_{\beta\gamma}\langle\omega^\beta|\wedge\langle\omega^\gamma|.\end{aligned} \tag{3.6}$$

The Poisson brackets of the non-canonical coordinates are easily calculated using the formulae of appendix 1:

$$\begin{array}{llll} \{z^i, z^j\}=0 & \{z^i, \zeta^\beta\}=0 & \{\zeta^\alpha, \zeta^\beta\}=0 & \{z^i, r_j\}=\delta^i{}_j \\ \{\zeta^\alpha, r_j\} = -R^\alpha{}_\beta(\zeta, 1)A_j^{\ \beta}(z) & \{z^i, r_\beta\}=0 & \{\zeta^\alpha, r_\beta\} = L^\alpha{}_\beta(\zeta, 1) \\ \{r_i, r_j\} = r_\alpha\ \mathrm{Ad}^\alpha{}_\beta(\zeta^{-1})F_{ij}^{\ \ \beta}(z) & \{r_i, r_\beta\}=0 & \{r_\alpha, r_\beta\} = -r_\gamma f^\gamma_{\alpha\beta}. \end{array} \tag{3.7}$$

† Abraham and Marsden (1978) and Woodhouse (1980) give proofs and a more detailed treatment of the general theory.

‡ The ket-bra notation is used here for vectors on phase space T^*B in order to distinguish them from the vector fields on configuration space B which were denoted in bold.

Due to the reparametrisation invariance of the action, the canonical Hamiltonian vanishes:

$$H_{\mathrm{can}}(P)=r_i v^i+r_\alpha v^\alpha-L=0 \tag{3.8}$$

and we have a primary first class constraint

$$K(P)\equiv g^{ij}(z)r_i r_j+(1/a^2)\eta^{\alpha\beta}r_\alpha r_\beta-m^2c^2=0. \tag{3.9}$$

The Hamiltonian of the system can be taken as

$$H(P)=F(u)K(P) \tag{3.10}$$

where $F(u)$ is an arbitrary function of the evolution parameter u. The equation of motion of a dynamical variable $O(P)$ is

$$\mathrm{d}O/\mathrm{d}u=\{O, H\}. \tag{3.11}$$

In particular we find

$$\mathrm{d}r_\alpha/\mathrm{d}u=\{r_\alpha, H\}=0. \tag{3.12}$$

The right group action R_γ on B, defined by (2.3), induces a left group action L_γ on phase space through its pull-back:

$$L_\gamma: T^*B\rightarrow T^*B: P=[Z,p]\rightarrow L_\gamma(P)=[Z\cdot\gamma^{-1}, R^*_\gamma|_{Z\cdot\gamma^{-1}}(p)]. \tag{3.13a}$$

In local coordinates (non-canonical!) this action reads

$$\begin{aligned} &z^i\rightarrow z^i \qquad && r_i\rightarrow r_i\\ &\zeta^\alpha\rightarrow(\zeta\cdot\gamma^{-1})^\alpha \qquad && r_\alpha\rightarrow r_\beta\,\mathrm{Ad}^\beta{}_\alpha(\gamma^{-1}). \end{aligned} \tag{3.13b}$$

The generating vector field of this group action

$$|E_\alpha(P)\rangle=\frac{\mathrm{d}}{\mathrm{d}t}L_{\gamma_\alpha(t)}{}^{(P)}|_{t=0}, \tag{3.14a}$$

is given in the basis (3.2) by

$$|E_\alpha\rangle=-(|e_\alpha\rangle+|\partial/\partial r_\beta\rangle r_\mu f^\mu_{\alpha\beta}). \tag{3.14b}$$

The symplectic potential $\langle\theta|$ is invariant under the above group action:

$$L^*{}_\gamma\langle\theta|=\langle\theta|, \tag{3.15}$$

so that we have an Ad^*-equivariant momentum mapping from T^*B to $\mathscr{L}^*(\mathrm{G})$, the dual of the Lie algebra of the group G:

$$J: T^*B\rightarrow\mathscr{L}^*(\mathrm{G}): P\rightarrow J(P)=\langle\theta|E_\alpha\rangle|_P\Delta^\alpha \tag{3.16}$$

where Δ^α is the basis of $\mathscr{L}^*(\mathrm{G})$ dual to the basis T_α of $\mathscr{L}(\mathrm{G})$. Furthermore the Hamiltonian vector fields corresponding to the dynamical variables $\langle\theta|E_\alpha\rangle=-r_\alpha$ are the vector fields $|E_\alpha\rangle$.

Fixing an element $\bar\rho$ of $\mathscr{L}^*(\mathrm{G})$, the restriction Σ' of Σ to the $2\times(4+N)-N$ dimensional manifold $J^{-1}(\bar\rho)$ defines a presymplectic structure on it:

$$\Sigma'=\Sigma_{|J^{-1}(\bar\rho)}=-\mathrm{d}\langle\theta|_{|J^{-1}(\bar\rho)}. \tag{3.17}$$

The kernel distribution of Σ', i.e. the distribution generated by the vector fields $|K\rangle$ of

$T_P(J^{-1}(\bar{\rho}))$ satisfying

$$\Sigma'[|K\rangle, |\ \rangle] = 0, \tag{3.18}$$

is integrable and defines a foliation Φ.

The solutions of (2.18) are given by $|K\rangle = |e_\alpha\rangle K^\alpha$, with

$$\bar{r}_\alpha f^\alpha_{\beta\gamma} K^\gamma = 0,$$

which is the infinitesimal form of the left action of the k-dimensional isotropy group $H(\bar{\rho})$ of $\bar{\rho} = -\bar{r}_\alpha \Delta^\alpha$†. The quotient manifold $J^{-1}(\bar{\rho})/\Phi = J^{-1}(\bar{\rho})/H(\bar{\rho})$, which has $2\times(4+N)-k$ dimensions, admits a unique symplectic 2-form Σ'' such that $\mathrm{proj}^*\,\Sigma'' = \Sigma'$, where proj is the natural projection $J^{-1}(\bar{\rho}) \to J^{-1}(\bar{\rho})/\Phi$.

Let $\mathscr{C}$ be a fixed Cartan subalgebra of $\mathscr{L}(\mathrm{G})$ with generators T_{α_0}, the other generators being T_{α_1}. Without loss of generality we may choose $\bar{\rho}$ as belonging to $\mathscr{C}$ so that the manifold $J^{-1}(\bar{\rho})$ is given by the N equations

$$r_{\alpha_0} = \bar{r}_{\alpha_0} \qquad \text{and} \qquad r_{\alpha_1} = 0. \tag{3.20}$$

The isotropy group $H(\bar{\rho})$ is then given by elements of the form

$$\gamma = \exp(\gamma^{\alpha_0} T_{\alpha_0}) \tag{3.21}$$

and under the left action of $H(\bar{\rho})$ points of $J^{-1}(\bar{\rho})$ transform as

$$z^i \to z^i \qquad r_i \to r_i \qquad \zeta^\alpha \to (\zeta\cdot\gamma^{-1})^\alpha \qquad \bar{r}_\alpha \to \bar{r}_\alpha. \tag{3.22}$$

Going to the quotient $J^{-1}(\bar{\rho})/\Phi$ amounts to projecting onto the equivalence classes of this action. Choosing as a representative of such an equivalence class the group element of the form

$$\zeta = \exp(\zeta^{\alpha_1} T_{\alpha_1})$$

coordinates on the reduced symplectic manifold are given by $(z^i, r_i, \zeta^{\alpha_1})$ and the reduced symplectic 2-form reads

$$\Sigma'' = \langle\omega^i| \wedge \langle \mathrm{d}r_i| - \tfrac{1}{2}\bar{r}_{\alpha_0}\Omega_{ij}{}^{\alpha_0}(Z)\langle\omega^i|\wedge\langle\omega^j| + \tfrac{1}{2}\bar{r}_{\alpha_0} f^{\alpha_0}_{\beta_1\gamma_1}\langle\omega^{\beta_1}|\wedge\langle\omega^{\gamma_1}|. \tag{3.23}$$

Besides the above outlined reduction, there is another reduction of phase space due to the constraint of (2.9). However, since $\{r_\alpha, K\} = 0$, this reduction is independent of the preceding one and the quantisation condition obtained in the next section is independent of it.

4. Charge quantisation

The pre-quantisation scheme of Kostant and Souriau, outlined in appendix 2, imposes the condition

$$\frac{1}{h}\int_C \theta = \nu \tag{4.1}$$

where ν is an integer and where C is a closed curve in $J^{-1}(\bar{\rho})$ contained in a leaf of the foliation Φ. In particular we may choose closed orbits $C(\alpha_0)$ of the Hamiltonian

† k is the dimension of a Cartan subalgebra of $\mathscr{L}(\mathrm{G})$. Note that $N-k=2p$ is even.

vector fields $-|E_{\alpha_0}\rangle_{|J^{-1}(\bar{\rho})} = |e_{\alpha_0}\rangle$ associated with the dynamical variables r_{α_0}†. These orbits are given by

$$z^i = z_0{}^i \qquad r_i = r_{0i} \qquad \zeta(\alpha_0;\, t) = \zeta_0 \cdot \gamma_{\alpha_0}(t) \tag{4.2}$$

where $\gamma_\alpha(t)$ was defined by (2.5).

The canonical 1-form along such an orbit is

$$\langle\theta|_{|C(\alpha_0)} = \bar{r}_{\alpha_0}\,\mathrm{d}t. \tag{4.3}$$

Let $\tau(\alpha_0)$ be the period of each $\zeta(\alpha_0;\, t)$:

$$\tau(\alpha_0) = \int_{C(\alpha_0)} \mathrm{d}t \tag{4.4}$$

then we obtain

$$\bar{r}_{\alpha_0} = h\nu(\alpha_0)/\tau(\alpha_0). \tag{4.5}$$

We consider the following cases of interest.

(1) Group U(1): $N = 1$, $k = 1$.

$$T_{\mathrm{em}} = -\mathrm{i} \qquad \tau_{\mathrm{em}} = 2\pi \qquad r_{\mathrm{em}} = h/2\pi\nu_{\mathrm{em}}.$$

(2) Group SU(2): $N = 3$, $k = 1$. Using the Pauli matrices we have $\alpha_0 = 3$ and obtain

$$T_3 = (1/2\mathrm{i})\sigma_3 \qquad \tau_3 = 4\pi \qquad r_3 = (h/4\pi)\nu_3.$$

(3) Group SU(3): $N = 8$, $k = 2$. With the Gell-Mann matrices we have $\alpha_0 = 3$ and 8, so that

$$T_3 = (1/2\mathrm{i})\lambda_3 \qquad \tau_3 = 4\pi \qquad r_3 = (h/4\pi)\nu_3$$

$$T_8 = (1/2\mathrm{i})\lambda_8 \qquad \tau_8 = 4\pi\sqrt{3} \qquad r_8 = (1/\sqrt{3})(h/4\pi)\nu_8.$$

The charges are thus quantised in terms of the elementary charge of a particle, given by‡

$$q = hc/2\pi e = 2\sqrt{\kappa}(h/2\pi ac). \tag{4.6}$$

We have

$$Q_{\alpha_0} = (c/e)r_{\alpha_0} = q(2\pi/\tau(\alpha_0))\nu(\alpha_0). \tag{4.7}$$

To examine more closely the range of allowed values of the integers $\nu(\alpha_0)$ we consider the van Hove operators (see appendix 2) $\hat{r}_\alpha$ corresponding to the variables r_α. These operators act on the Hilbert space of the square integrable functions (with Liouville measure) on phase space:

$$\hat{r}_\alpha\psi = \mathrm{i}(h/2\pi)E_\alpha[\psi]. \tag{4.8}$$

As operators on this Hilbert space $\mathscr{h}$ they obey the commutation relations

$$[\hat{r}_\alpha, \hat{r}_\beta] = -\mathrm{i}(h/2\pi)f^\gamma_{\alpha\beta}\hat{r}_\gamma \tag{4.9}$$

† α_0 is an index of the fixed Cartan subalgebra $\mathscr{C}$.

‡ There is a fundamental unit of length for each simple component of the structure group, with corresponding independent units of charge.

and they generate the group representation in $\mathscr{h}$:

$$D(\gamma):\mathscr{h}\to\mathscr{h}:\psi(P)\to(D(\gamma)\psi)(P)=\psi(L_\gamma^{-1}P), \tag{4.10}$$

where L_γ is the left group action on phase space defined in (3.13).

In terms of the variables ζ^α and r_α the Liouville measure is just the product of the bi-invariant Haar measure on the group manifold with the cartesian measure in r_α space†. Since $\det[\mathrm{Ad}(\zeta)]=1$, this measure is invariant under the group action and the representation is unitary. By general theorems on the representations of compact groups it follows that the representation is reducible to a sum of irreducible unitary representations, each corresponding to definite values of the Casimir operators of the Lie algebra. Within each irreducible representation the integers $\nu(\alpha_0)$ will vary in a well defined range.

5. Conclusion

Thus, we have shown that the pre-quantisation condition (4.1) is sufficient to obtain charge quantisation. This is not surprising because we already knew that, in Klein's quantum approach, the charge is quantised, and on the other hand the pre-quantisation condition (3.1) is precisely what is needed in the classical theory to obtain the quantisation condition of the Bohr-Wilson-Sommerfeld type as shown in appendix 2.

It can also be shown that the mass is quantised in the following sense. The mass m' ('dressed' mass) in the Lagrangian formalism is related to the parameter m of the theory by (1.34), which can be rewritten as $m'^2=m^2-(1/ac)^2\eta^{\alpha\beta}r_\alpha r_\beta$. This is nothing other than the constraint (2.9) in the Hamiltonian formalism with $m'^2c^2=g^{ij}r_ir_j$. Denoting $C_2=-(2\pi/h)^2\eta^{\alpha\beta}r_\alpha r_\beta$‡, we obtain

$$m'^2=m^2+(h/2\pi ac)^2C_2$$

with, for the cases considered in § 4,

$$C_2=\nu_{\mathrm{em}}^2 \qquad \text{for U(1)}$$

$$C_2=\tfrac{1}{2}(\tfrac{1}{2}\nu_3)^2 \qquad \text{for SU(2) and}$$

$$C_2=\tfrac{1}{3}[(\tfrac{1}{2}\nu_3)^2+(\tfrac{1}{2}(\tfrac{1}{3})^{1/2}\nu_8)^2] \qquad \text{for SU(3).}$$

In Klein's approach one had $m=0$ so that $m'=(h/2\pi ac)\sqrt{C_2}$.

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† We do not have to consider the z^i, r_i dependence of the functions on phase space since these coordinates are not affected by the group action.

‡ Group theoretically C_2 is the quadratic Casimir invariant.

Appendix 1: Lagrangian and Hamiltonian formalism in non-canonical coordinates

A system of coordinates x^i of points X of configuration space Q induces holonomic bases of tangent and cotangent spaces, T_XQ and T_X^*Q respectively, denoted by $\boldsymbol{\partial}/\partial x^i$ and $\mathrm{d}\boldsymbol{x}^i$.

Let $\boldsymbol{e}_\alpha$ and $\boldsymbol{\varepsilon}^\alpha$ denote a dual pair of anholonomic bases with structure functions defined by

$$[\boldsymbol{e}_\alpha, \boldsymbol{e}_\beta] = c^\gamma_{\alpha\beta}(X)\boldsymbol{e}_\gamma \tag{A1.1a}$$

or

$$\mathrm{d}\boldsymbol{\varepsilon}^\alpha = -\tfrac{1}{2}c^\alpha_{\beta\gamma}(X)\boldsymbol{\varepsilon}^\beta \wedge \boldsymbol{\varepsilon}^\gamma. \tag{A1.1b}$$

The Lagrangian is defined as a function on the tangent bundle TQ with points $X' = (X, \boldsymbol{v})$ which can be given by the coordinates $(x^i, u^i = \mathrm{d}\boldsymbol{x}^i \cdot \boldsymbol{v})$ or by the non-canonical coordinates $(x^i, v^\alpha = \boldsymbol{\varepsilon}^\alpha \cdot \boldsymbol{v})$ and we have two different functions of these coordinates:

$$\mathrm{Lagr}(X') = L_c(x^i, u^i) = L(x^i, v^\alpha). \tag{A1.2}$$

The momentum is given by

$$p_i = \partial L_c/\partial u^i = \boldsymbol{p} \cdot \boldsymbol{\partial}/\partial x^i, \tag{A1.3a}$$

or by

$$r_\alpha = \partial L/\partial v^\alpha = \boldsymbol{p} \cdot \boldsymbol{e}_\alpha. \tag{A1.3b}$$

The Euler–Lagrange equations read

$$\mathrm{d}p_i/\mathrm{d}t - \partial L_c/\partial x^i = 0 \tag{A1.4a}$$

or

$$\mathrm{d}r_\alpha/\mathrm{d}t + r_\mu c^\mu_{\alpha\beta}(X)v^\beta - e_\alpha(L) = 0 \tag{A1.4b}$$

where $\boldsymbol{v} = \mathrm{d}X/\mathrm{d}t$.

The points $X'' = (X, \boldsymbol{p})$ of phase space T^*Q have canonical and non-canonical coordinates, (x^i, p_i) and (x^i, r_α) respectively, with corresponding holonomic bases of $T_{X''}(T^*Q)$ and $T^*_{X''}(T^*Q)$ given by

$$(|\partial/\partial x^i\rangle_c, |\partial/\partial p_i\rangle_c) \qquad (|\partial/\partial x^i\rangle_{nc}, |\partial/\partial r_\alpha\rangle_{nc}) \tag{A1.5a}$$

with duals

$$(\langle \mathrm{d}x^i|_c, \langle \mathrm{d}p_i|_c) \qquad (\langle \mathrm{d}x^i|_{nc}, \langle \mathrm{d}r_\alpha|_{nc}). \tag{A1.5b}$$

They are related by:

$$|\partial/\partial x^i\rangle_c = |\partial/\partial x^i\rangle_{nc} + r_\alpha \varepsilon^\alpha{}_j\, \partial_i e^j{}_\beta |\partial/\partial r_\beta\rangle_{nc} \tag{A1.6a}$$

$$|\partial/\partial p_i\rangle_c = e^i{}_\alpha |\partial/\partial r_\alpha\rangle_{nc} \tag{A1.6b}$$

and

$$\langle \mathrm{d}x^i|_c = \langle \mathrm{d}x^i|_{nc} \tag{A1.7a}$$

$$\langle \mathrm{d}p_i|_c = \langle \mathrm{d}r_\alpha|_{nc}\varepsilon^\alpha{}_i + r_\alpha\, \partial_j \varepsilon^\alpha{}_i \langle \mathrm{d}x^j|_{nc}. \tag{A1.7b}$$

Useful anholonomic bases of $T_{X''}(T^*Q)$ and $T^*_{X''}(T''Q)$ are

$$|e_\alpha\rangle = |\partial/\partial x^i\rangle_{\mathrm{nc}} e^i{}_\alpha(X) \tag{A1.8a}$$

$$|\partial/\partial r_\alpha\rangle = |\partial/\partial r_\alpha\rangle_{\mathrm{nc}} \tag{A1.8b}$$

with dual

$$\langle \varepsilon^\alpha| = \varepsilon^\alpha{}_i(X)\langle \mathrm{d}x^i|_{\mathrm{nc}} \tag{A1.9a}$$

$$\langle \mathrm{d}r_\alpha| = \langle \mathrm{d}r_\alpha|_{\mathrm{nc}}. \tag{A1.9b}$$

The symplectic potential, 1-form on T^*Q, reads

$$\langle\theta| = p_i\langle \mathrm{d}x^i| = r_\alpha\langle\varepsilon^\alpha| \tag{A1.10}$$

and the symplectic 2-form is obtained as

$$\Sigma = -\mathrm{d}\langle\theta| = \langle \mathrm{d}x^i|_{\mathrm{c}} \wedge \langle \mathrm{d}p_i|_{\mathrm{c}} \tag{A1.11a}$$

$$= \langle\varepsilon^\alpha| \wedge \langle \mathrm{d}r_\alpha| + \tfrac{1}{2} r_\mu c^\mu_{\alpha\beta}(X)\langle\varepsilon^\alpha| \wedge \langle\varepsilon^\beta|. \tag{A1.11b}$$

The Hamiltonian vector field $|H(F)\rangle$ associated with a classical observable

$$F(X'') = f_{\mathrm{c}}(x^i, p_i) = f(x^i, r_\alpha) \tag{A1.12}$$

is defined by:

$$\Sigma[|H(F)\rangle, |\ \rangle] = \langle \mathrm{d}F|\ \rangle. \tag{A1.13}$$

In coordinates it is given by

$$|H(F)\rangle = |\partial/\partial x^i\rangle_{\mathrm{c}}\, \partial f_{\mathrm{c}}/\partial p_i - |\partial/\partial p_i\rangle_{\mathrm{c}}\, \partial f_{\mathrm{c}}/\partial x^i \tag{A1.14a}$$

$$= |e_\alpha\rangle\, \partial f/\partial r_\alpha - |\partial/\partial r_\alpha\rangle(e_\alpha(f) + r_\mu c^\mu_{\alpha\beta}(X)\, \partial f/\partial r_\beta). \tag{A1.14b}$$

The Poisson brackets of two observables are

$$\{F, G\} = \Sigma[|H(F)\rangle, |H(G)\rangle] \tag{A1.15}$$

$$= \partial f_{\mathrm{c}}/\partial x^i\, \partial g_{\mathrm{c}}/\partial p_i - \partial f_{\mathrm{c}}/\partial p_i\, \partial g_{\mathrm{c}}/\partial x^i \tag{A1.16a}$$

$$= e_\alpha(f)\, \partial g/\partial r_\alpha - \partial f/\partial r_\alpha e_\alpha(g) - r_\mu c^\mu_{\alpha\beta}(X)\, \partial f/\partial r_\alpha\, \partial g/\partial r_\beta. \tag{A1.16b}$$

Appendix 2. Pre-quantisation

The pre-quantisation scheme of Kostant and Souriau (see, for example, Woodhouse 1980) is a globalisation of the construction, due to van Hove (1951), of an isomorphism of the Lie algebra of the classical observables with the Poisson bracket operation, on the Lie algebra of symmetric operators on the Hilbert space of the square integrable functions on phase space with the commutator operation.

Consider a principal U(1) bundle E over a symplectic manifold S with its symplectic structure given by the 2-form σ. Points e of E will be given in a local gauge by $e \to [x, \exp(\mathrm{i}\xi)]$, where x is a point of S. The bundle projection is denoted by

$$Pr: E \to S: e \to x = Pr(e). \tag{A2.1}$$

The group U(1) acts on the right on E and equivariant functions on E are defined such that

$$\Psi[e\cdot\exp(\mathrm{i}\alpha)]=\exp(-\mathrm{i}\alpha)\Psi[e]. \tag{A2.2}$$

In a local gauge they have the following form

$$\Psi[x,\xi]=\exp(-\mathrm{i}\xi)\psi(x). \tag{A2.3}$$

A scalar product of such functions is defined by

$$(\Psi,\Phi)=h^{-n}\int_E \Psi^*[e]\Phi[e]\mu_{\mathrm{L}}\otimes\frac{\mathrm{d}\xi}{2\pi}, \tag{A2.4}$$

where μ_{L} is the Liouville measure on the $2n$-dimensional phase space S and where h will be identified with Planck's constant. The square integrable functions on E with the above product form a Hilbert space $\hbar$.

The bundle E has a connection given in a local gauge by

$$\alpha=-(2\pi/h)\theta+\mathrm{d}\xi \tag{A2.5}$$

where θ is a local symplectic potential so that the curvature of the connection equals the pull-back of the symplectic 2-form σ:

$$\Omega=Pr^*(2\pi\sigma/h). \tag{A2.6}$$

Weil's integrality condition states that for such a bundle to exist it is necessary and sufficient that the closed 2-form $\Omega/2\pi$ should define an integral cohomology class of S. This condition is trivially satisfied when the symplectic manifold S is a cotangent bundle T^*Q since E is then a trivial bundle and globally $\Omega=\mathrm{d}\alpha$ is exact. It is however not trivial when S arises from the reduction of a cotangent bundle and it can be shown (Woodhouse 1980) that Weil's integrality condition turns into the condition that

$$1/h\int_C \theta=I(C) \tag{A2.7}$$

must be integer valued when C is a closed orbit belonging to a leaf of the foliation defined by the pre-symplectic 2-form σ' which is the restriction of σ to the constrained submanifold of S.

A more pedestrian way to see how this condition arises is based upon the construction (van Hove 1950) of a symmetric operator, associated with each classical observable, acting on $\hbar$. To $A(x)$ we associate $\hat{A}:\hbar\to\hbar$ defined by

$$\hat{A}\Psi=-\mathrm{i}(h/2\pi)\nabla_{H(A)}\Psi+A\Psi \tag{A2.8}$$

where ∇ is the covariant derivative of equivariant functions, defined by the connection α and where $H(A)$ is the Hamiltonian vector field associated with $A(x)$.

In a local gauge where the function $\Psi[e]$ is given by $\psi(x)$, the operator $\hat{A}$ acts as

$$\hat{A}\psi=-\mathrm{i}(h/2\pi)H(A)[\psi]+[A-(\theta,H(A))]\psi. \tag{A2.9}$$

Considering the eigenvalue equation

$$\hat{A}\psi=a\psi. \tag{A2.10}$$

we notice that, on the orbits of the Hamiltonian vector field $H(A)$, the function $A(x)$ remains constant and that

$$-\mathrm{i}(h/2\pi)H(A)[\psi]-(\theta,H(A))\psi=0$$

or $\mathrm{d}\psi/\psi = 2\pi \mathrm{i}\theta/h$, which yields

$$\psi(x(t)) = \exp\left(\frac{2\pi\mathrm{i}}{h}\int_{C(t,t_0)}\theta\right)\psi(x(t_0)). \tag{A2.11}$$

It follows that for a closed orbit $C(t, t_0)$ this implies the Bohr-Wilson-Sommerfeld condition.

Appendix 3. On the ansatz for the metric and the action

In order to implement spontaneous fibration in the sense that the metric be a possible solution of all of the $(4+N)$-dimensional Einstein's equations, it is necessary to consider more general metrics than that of (1.12). This amounts to the non-Abelian generalisation (Cho and Freund 1975) of the Jordan-Thiry theory (see, for example, Lichnerowicz 1955) rather than of the original Kaluza-Klein. The metric

$$g(X) = g_{ij}(\pi(X))\omega^i\otimes\omega^j + H_{\alpha\beta}(X)\omega^\alpha\otimes\omega^\beta \tag{A3.1}$$

is still right invariant if

$$H_{\alpha\beta}(X\cdot\gamma) = H_{\mu\nu}(X)\,\mathrm{Ad}^\mu{}_\alpha(\gamma)\,\mathrm{Ad}^\nu{}_\beta(\gamma), \tag{A3.2}$$

so that in a local gauge

$$H_{\alpha\beta}(x,\xi) = h_{\mu\nu}(x)\,\mathrm{Ad}^\mu{}_\alpha(\xi)\,\mathrm{Ad}^\nu{}_\beta(\xi). \tag{A3.3}$$

Due to this right invariance, the Riemannian curvature scalar $\mathscr{R}(X)$ will be still constant on each fibre. This implies that in a generalised Hilbert type action we may integrate over the group volume and obtain an effective action in four-dimensional space. It can be checked that the obtained equations of motion are the same as those to which the $(4+N)$-dimensional Einstein's equations reduce to when the ansatz metric (A3.1) is substituted in them. The Euler-Lagrange equations for the particle's motion are†

$$\frac{\mathrm{d}}{\mathrm{d}u}r_i - \left\{ {p \atop iq} \right\} r_p v^q = r_\alpha \Omega_{ij}{}^\alpha(Z)v^j + \tfrac{1}{2}r_\alpha\,\mathrm{Ad}^\alpha{}_\mu(\zeta^{-1})h^{\mu\rho}\nabla_i h_{\rho\nu}\,\mathrm{Ad}^\nu{}_\beta(\zeta)v^\beta \tag{A3.4}$$

and

$$\mathrm{d}r_\alpha/\mathrm{d}u = 0. \tag{A3.5}$$

The pre-quantisation condition (3.1) will again yield the quantisation of the momenta r_α, but the physical interpretation will be more complicated. For the Abelian Jordan-Thiry theory we refer to Lichnerowicz (1955).

We have not considered such a generalisation mainly for simplicity reasons but also because we are somewhat reluctant to introduce $N(N+1)/2$ additional scalar fields $h_{\alpha\beta}(x)$ whose interpretation as candidates for Higgs fields seems questionable. Naturally the theory we consider is then not an example of spontaneous fibration in the sense discussed above.

† The field equations are discussed in Cho and Freund (1975), Orzalesi (1981) and in Coquereaux and Jadczyk (1983).

References

Abraham R and Marsden J E 1978 *Foundations of Mechanics* (Reading, Mass.: Benjamin Cummings)
Cho Y M 1975 *J. Math. Phys.* **16** 2029
Cho Y M and Freund P G O 1975 *Phys. Rev.* D **12** 1711
Coqueraux R 1983 *CERN Preprint Th-3639*
Coqueraux R and Jadczyk A 1983 *Commun. Math. Phys.* **90** 79
Duval C 1981 *Ann. Inst. H. Poincaré* **34** 95
Duval C and Horvathy P 1982 *Ann. Phys., NY* **142** 10
Kaluza Th 1921 *Sitz. Preuss. Akad. Wiss.* 966
Klein O 1926 *Z. Phys.* **37** 895
Lichnerowicz A 1955 *Théories relativistes de la gravitation et de l'électromagnétisme* (Paris: Hasson)
Orzalesi C A 1981 *Fortschr. Phys.* **29** 413
Orzalesi C A and Pauri M 1982 *Nuovo Cimento* B **68** 193
Salam A and Strathdee J 1982 *Ann. Phys., NY* **141** 316
Vanhecke F J 1982 *Rev. Bras. Fís.* **12** 343
van Hove L 1951 *Mém. Acad. R. Belg. Coll. Sci.* **t26** 61
Weinberg S 1983 *Phys. Lett.* **125B** 265
Witten E 1981 *Nucl. Phys.* B **186** 412
Woodhouse N 1980 *Geometric Quantization* (Oxford: Clarendon)